A Unified Algorithm of Finite Element Method for Circuit Coupled 2-D Plane and Axisymmetric Magnetic Fields and Its Applications

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Abstract —A formulation of finite element methods (FEM) for both two-dimensional (2-D) plane and axisymmetric magnetic fields - electric circuit co-simulation is presented. The advantage is that a unified formulation is realized for both plane and axisymmetric fields which are coupled with arbitrarily connected electric circuits.

I. INTRODUCTION

Finite element methods (FEM) of magnetic field computation have been widely used in electrical engineering. Many problems can be simplified as the study of two-dimensional (2-D) plane field and axisymmetric field. If field distributions have axial symmetry, axisymmetric FEM is much more efficient compared with three-dimensional (3-D) FEM. The basic formulation of axisymmetric magnetic field has been reported in [1-2]. For voltage fed devices, the current can be computed from the magnetic field simultaneously [3]. Because windings and conductors are usually connected with other electric circuits, magnetic field and electric circuit coupled methods are commonly required [4-5]. However, there are hardly any publications detailing the general formulations in both stranded winding and solid conductor regions for transient analysis of axisymmetric magnetic field which are coupled with arbitrarily connected external electric circuit.

A systematic method to deduce the formulation of both 2-D plane and axisymmetric FEM for magnetic field - arbitrary connected electric circuit co-simulation is presented in this paper. A unified general formulation is achieved. The coupling ports can be stranded windings and solid conductors.

II. UNIFIED MAGNETIC FIELD FORMULATION FOR AXISYMMETRIC PROBLEM

The basic magnetic field equation for 2-D plane problem and axisymmetric problem can be summarized in one equation:

$$lp\nabla\times(\nu\nabla\times\boldsymbol{A})+lp\sigma\frac{\partial\boldsymbol{A}}{\partial t}-\frac{ld_pN_w}{S_wa}i_w\boldsymbol{a}_w-\frac{pd_pa\sigma}{N_w}u_w\boldsymbol{a}_w=lp\boldsymbol{J}_s+lp\nabla\times\boldsymbol{H}_c, \quad (1)$$

where l has more general meaning, which is the depth of the model l_z for 2-D plane problem and is equal to $2\pi r$ for axisymmetric problem. lp is multiplied to each term in (1) such that the integral of each term on the u-v plane means the integral on the volume. It is in line with real problems in 3-D space. If the currents $\boldsymbol{i}_w$ in stranded windings are known and the voltage differences $\boldsymbol{u}_w$ on the solid conductors are known, the magnetic field distribution can be obtained by solving the magnetic field equation (1). Using Galerkin method and using the shape function $\boldsymbol{N}$ as the weighting function,

$$\iint_\Omega lp(\nabla\times\boldsymbol{A})^T\cdot\nu\nabla\times\boldsymbol{N}\mathrm{d}\Omega+\iint_\Omega lp\sigma\frac{\partial\boldsymbol{A}}{\partial t}\cdot\boldsymbol{N}\mathrm{d}\Omega-\iint_\Omega\frac{ld_pN_w}{S_wa}i_w\boldsymbol{a}_w\cdot\boldsymbol{N}\mathrm{d}\Omega$$
$$-\iint_\Omega\frac{pd_pa\sigma}{N_w}u_w\boldsymbol{a}_w\cdot\boldsymbol{N}\mathrm{d}\Omega=\iint_\Omega lp\boldsymbol{J}_s\cdot\boldsymbol{N}\mathrm{d}\Omega+\iint_\Omega lp(\nabla\times\boldsymbol{H}_c)\cdot\boldsymbol{N}\mathrm{d}\Omega. \quad (2)$$

In 2-D plane and axisymmetric problems, $\boldsymbol{A}$ and $\boldsymbol{N}$ only have components perpendicular to the solution plane u-v. If the scalar variables A and N are the respective components in the $\boldsymbol{a}_w$ direction:

$$A=\sum_{k=1}^{6}N_k(x,y)A_k=[\boldsymbol{N}]^T[\tilde{\boldsymbol{A}}], \quad (3)$$

For 2-D plane problem:

$$\nabla\times\boldsymbol{A}=\left(\frac{\partial A_z}{\partial y}\boldsymbol{a}_x-\frac{\partial A_z}{\partial x}\boldsymbol{a}_y\right). \quad (4)$$

For axisymmetric problems:

$$\nabla\times\boldsymbol{A}=\begin{bmatrix}-\dfrac{\partial A_\varphi}{\partial z}\\ \dfrac{\partial A_\varphi}{\partial r}+\dfrac{A_\varphi}{r}\end{bmatrix}. \quad (5)$$

For 2-D plane problems:

$$\nabla\times\boldsymbol{N}_i=\begin{bmatrix}\dfrac{\partial N_i}{\partial y}\\ -\dfrac{\partial N_i}{\partial x}\end{bmatrix}=\begin{bmatrix}(\nabla N_i)_y\\ -(\nabla N_i)_x\end{bmatrix}=\begin{bmatrix}(\mathrm{rot}\boldsymbol{N}_i)_x\\ (\mathrm{rot}\boldsymbol{N}_i)_y\end{bmatrix}. \quad (6)$$

For axisymmetric problems:

$$\nabla\times\boldsymbol{N}_i=\begin{bmatrix}-(\nabla N_i)_z\\ (\nabla N_i)_r+\dfrac{N_i}{r}\end{bmatrix}=\begin{bmatrix}(\mathrm{rot}\boldsymbol{N}_i)_r\\ (\mathrm{rot}\boldsymbol{N}_i)_z\end{bmatrix}. \quad (7)$$

In (6) and (7), $\boldsymbol{N}_i$ is a vector in the $\boldsymbol{a}_w$ direction.

III. FORMULATION FOR CIRCUIT COUPLING

If stranded windings and solid conductors in the magnetic field regions are connected with an external arbitrarily connected electric circuit, the currents $\boldsymbol{i}_w$ in stranded windings and the voltage differences $\boldsymbol{u}_w$ on solid conductors are unknown. The magnetic field equations should be coupled with the circuit equations when deriving the magnetic field distribution and currents/voltages in the electric circuit. Here the circuit equations are established using the loop method. The unknowns in the circuit part are the loop currents.

The basic equations in the regions of stranded windings and solid conductors can be summarized as:

$$\begin{cases}lp\nabla\times(\nu\nabla\times\boldsymbol{A})+lp\sigma\dfrac{\partial\boldsymbol{A}}{\partial t}-\dfrac{d_pN_w}{a\iint_\Omega\frac{1}{l}\mathrm{d}\Omega}(i_w+i_{ad})\boldsymbol{a}_w=0\\ -\dfrac{N_w}{a\iint_\Omega\frac{1}{l}\mathrm{d}\Omega}\iint_\Omega d_p\dfrac{\partial A}{\partial t}\mathrm{d}\Omega-R_wi_w=-u_w\\ -\dfrac{N_w}{S_wa}\iint_\Omega d_pl\dfrac{\partial A}{\partial t}\mathrm{d}\Omega+R_wi_{ad}=0.\end{cases} \quad (8)$$

The field and circuit branch equations in the magnetic field region can be written in the matrix format:

$$\begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \boldsymbol{C}_{13} \\ 0 & \boldsymbol{C}_{22} & 0 \\ 0 & 0 & \boldsymbol{C}_{33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{A} \\ \boldsymbol{i}_w \\ \boldsymbol{i}_{ad} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{D}_{11} & 0 & 0 \\ \boldsymbol{D}_{21} & 0 & 0 \\ \boldsymbol{D}_{31} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\boldsymbol{i}_w}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\boldsymbol{i}_{ad}}{\mathrm{d}t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\boldsymbol{u}_w \\ 0 \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{P}_A \\ 0 \\ 0 \end{Bmatrix}, \quad (9)$$

where $\boldsymbol{D}_{21} = \boldsymbol{C}_{12}^T \quad \boldsymbol{D}_{31} = \boldsymbol{C}_{13}^T$.

Using the backward Euler's method to discretize the time variable and multiply Δt to the additional equation and the branch equation, one obtains the recurrence formulas:

$$\begin{bmatrix} \boldsymbol{C}_{11} + \dfrac{\boldsymbol{M}_{11}}{\Delta t} & \boldsymbol{C}_{12} & \boldsymbol{C}_{13} \\ \boldsymbol{D}_{21} & \Delta t \boldsymbol{C}_{22} + \boldsymbol{D}_{22} & 0 \\ \boldsymbol{D}_{31} & 0 & \Delta t \boldsymbol{C}_{33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{A}^k \\ \boldsymbol{i}_w^k \\ \boldsymbol{i}_{ad}^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\Delta t \boldsymbol{u}_w^k \\ 0 \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{P}_A + \dfrac{\boldsymbol{D}_{11}}{\Delta t} \boldsymbol{A}^{k-1} \\ \boldsymbol{D}_{21} \boldsymbol{A}^{k-1} \\ \boldsymbol{D}_{31} \boldsymbol{A}^{k-1} \end{Bmatrix}. \quad (10)$$

The branch equation of the external circuits can be expressed as:

$$[\boldsymbol{R}_e]\{\boldsymbol{i}_e\} = \{\boldsymbol{u}_e\} + \{\boldsymbol{P}_e\}, \quad (11)$$

where $\boldsymbol{R}_e$ is the matrix of the resistance and $\boldsymbol{P}_e$ is the column matrix associated with sources. By coupling the external circuit, the system equation can be derived as:

$$\begin{bmatrix} \boldsymbol{C}_{11} + \dfrac{\boldsymbol{D}_{11}}{\Delta t} & \boldsymbol{C}_{12} & 0 & \boldsymbol{C}_{13} \\ \boldsymbol{D}_{21} & \Delta t \boldsymbol{C}_{22} + \boldsymbol{D}_{22} & 0 & 0 \\ 0 & 0 & -\Delta t \boldsymbol{R}_e & 0 \\ \boldsymbol{D}_{31} & 0 & 0 & \Delta t \boldsymbol{C}_{33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{A}^k \\ \boldsymbol{i}_w^k \\ \boldsymbol{i}_e^k \\ \boldsymbol{i}_{ad}^k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\Delta t \boldsymbol{u}_f^k \\ -\Delta t \boldsymbol{u}_e^k \\ 0 \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{P}_A + \dfrac{\boldsymbol{D}_{11}}{\Delta t} \boldsymbol{A}^{k-1} \\ \boldsymbol{D}_{21} \boldsymbol{A}^{k-1} \\ -\Delta t \boldsymbol{P}_e \\ \boldsymbol{D}_{31} \boldsymbol{A}^{k-1} \end{Bmatrix}. \quad (12)$$

The connection of arbitrary connected electric circuits can be expressed by an incidence matrix. Using the loop method, the relationship between the branch current $\boldsymbol{i}_w$ and the loop current $\boldsymbol{i}_l$ is:

$$\{\boldsymbol{i}_w\} = [\boldsymbol{B}_{lb}^T]\{\boldsymbol{i}_l\}, \quad (13)$$

where $\boldsymbol{B}_{lb}$ is the loop-to-branch incidence matrix. The Kirchhoff's voltage law can be expressed as,

$$[\boldsymbol{B}_{lb}]\{\boldsymbol{u}_w\} = 0. \quad (14)$$

Substituting these relationships into the system equations, we obtain the final global equations:

$$\begin{bmatrix} \boldsymbol{C}_{11} + \dfrac{\boldsymbol{D}_{11}}{\Delta t} & \begin{pmatrix} \boldsymbol{C}_{12} & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{B}_{lb}^T & \boldsymbol{C}_{13} \\ \boldsymbol{B}_{lb} \begin{pmatrix} \boldsymbol{D}_{21} & 0 \\ 0 & 0 \end{pmatrix} & \boldsymbol{B}_{lb} \begin{pmatrix} \Delta t \boldsymbol{C}_{22} + \boldsymbol{D}_{22} & 0 \\ 0 & -\Delta t \boldsymbol{R}_e \end{pmatrix} \boldsymbol{B}_{lb}^T & 0 \\ \boldsymbol{D}_{31} & 0 & \Delta t \boldsymbol{C}_{33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{A}^k \\ \boldsymbol{i}_l^k \\ \boldsymbol{i}_{ad}^k \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_A + \dfrac{\boldsymbol{D}_{11}}{\Delta t} \boldsymbol{A}^{k-1} \\ \boldsymbol{B}_{lb} \begin{pmatrix} \boldsymbol{D}_{21} \boldsymbol{A}^{k-1} \\ -\Delta t \boldsymbol{P}_e \end{pmatrix} \\ \boldsymbol{D}_{31} \boldsymbol{A}^{k-1} \end{Bmatrix}, \quad (15)$$

where the coefficient matrix is symmetrical.

IV. EXAMPLES

The aforementioned methods have been used to simulate electric devices. Here an electrodynamic levitation device in TEAM Workshop Problem No. 28 is used as an axisymmetric example [6]. To test the proposed unified method, we intentionally connect the windings as external circuit. The measured stable levitation height is 11.3 mm and the computed is 11.45 mm, as shown in Fig.1. The numerical error is 1.3%. The computed result has very good agreement with measurement.

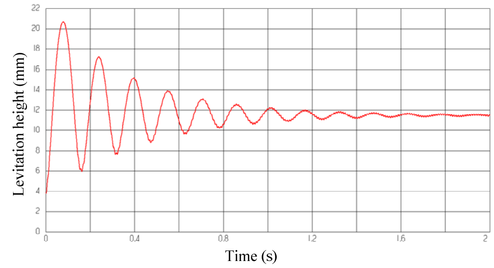

Fig. 1. Computed levitation height

A practical application of the axisymmetric FEM for designing a novel tubular linear PM (TLPM) motor is presented in this paper. The steady-state and transient performances of the TLPM machine are calculated. The configuration is shown in Fig. 2. By using the axisymmetric FEM, the magnetic flux line distribution under full load is shown in Fig. 3. It can be observed that the flux lines are smooth and most of the flux lines penetrate through both the flux-modulation poles and the two airgaps. This structure can effectively minimize the flux leakage, while the torque and power are transmitted effectively between the low-speed and high-speed movers. The calculation results consist with the operation principle of the machine.

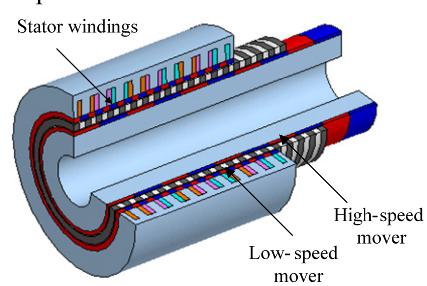

Fig.2. Configuration of the magnetic gear integrated brushless PM machine.

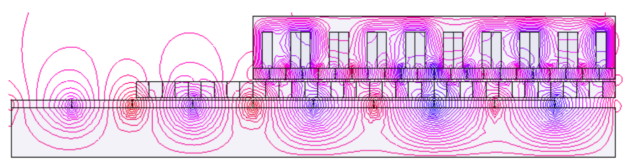

Fig.3. Magnetic flux line at full load.

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